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AN ASYMPTOTIC ANALYSIS OF THE SOLUTION IN THE NEIGHBOURHOOD OF THE CORNER POINT OF A CRACK ALONG THE KINKED INTERFACE OF TWO MEDIA†

A. V. ANDREYEV, R. V. GOL'DSHTEIN and Yu. V. ZHITNIKOV

Moscow

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The asymptotic behaviour of an elastic field in the neighbourhood of the corner point of a crack at the interface of different materials is investigated within the framework of plane elasticity, taking into account the contact of its surfaces and the possibility of their mutual slippage with dry friction. The problem is solved by the method of complex Kolosov–Muskhelishvili potentials. The results obtained enable one to estimate the angular range of existence of contact zones and the singularity of the stresses close to the corner point of the crack. It is shown that the formation of contact zones, taking into account the friction forces accompanying slippage, depends essentially on the magnitude of the angle of the interface kinking the elasticity moduli of the materials and the friction coefficient. Numerical calculations are carried out and the stress and displacement distributions in the neighbourhood of the corner point are obtained.

It is well known [1, 2] that the formation of contact zones along a crack has a considerable effect on the magnitude of the stress intensity factor. The asymptotic form of the solution in the neighbourhood of the vertex angle has been obtained in a homogeneous material [3, 4] and for a composite wedge [5].

1. FORMULATION OF THE PROBLEM

A two-dimensional problem of the linear elasticity on the distribution of the stresses an displacements in the neighbourhood of the corner point of a crack at the kinked interface of two media (Fig. 1) is considered, taking into account formation of a contact zone of its surfaces which is adjacent to the corner point of the crack. The following versions of the behaviour of the solution are possible: (1) no contact zone is formed in the neighbourhood of the corner point, that is, the crack surfaces are free; (2) a contact zone forms in a part of the crack boundary, adjacent to the corner point; (3) the whole crack surface in the neighbourhood of the corner point is in contact.

In the first case, the asymptotic behaviour of the solution is divided into two independent asymptotic forms for the homogeneous materials for angles adjacent to the corner point, respectively. These asymptotic forms are known [3, 4] and will not be considered here.

Contact interaction between the crack surfaces occurs in the last two cases and these will be considered below.

We will now formulate the boundary-value problem for the second case. Suppose the boundary $\theta = \alpha$ corresponds to the free surface and $\theta = 0$ corresponds to the contact zone between the crack surfaces adjacent to the boundary of the fracture (r and θ constitute a system of polar coordinates, associated with the corner point of the crack and α is the kinking angle of the interface). The boundary-value problem, taking into account the forces of dry friction in the contact zone has the form

$$\sigma_{\theta\theta}^{(1)}(r,\alpha) = \sigma_{\theta\theta}^{(2)}(r,\alpha) = 0, \quad \sigma_{r\theta}^{(1)}(r,\alpha) = \sigma_{r\theta}^{(2)}(r,\alpha) = 0 \tag{1.1}$$

$$u_\theta^{(1)}(r,0) = u_\theta^{(2)}(r,2\pi), \quad \sigma_{\theta\theta}^{(1)}(r,0) = \sigma_{\theta\theta}^{(2)}(r,2\pi) \tag{1.2}$$

$$\sigma_{r\theta}^{(1)}(r,0) = \rho\sigma_{\theta\theta}^{(1)}(r,0)\frac{(u_r^{(1)}(r,0) - u_r^{(2)}(r,2\pi))}{|(u_r^{(1)}(r,0) - u_r^{(2)}(r,2\pi))|} \tag{1.3}$$

$$\sigma_{r\theta}^{(1)}(r,0) = -\sigma_{r\theta}^{(2)}(r,2\pi)$$

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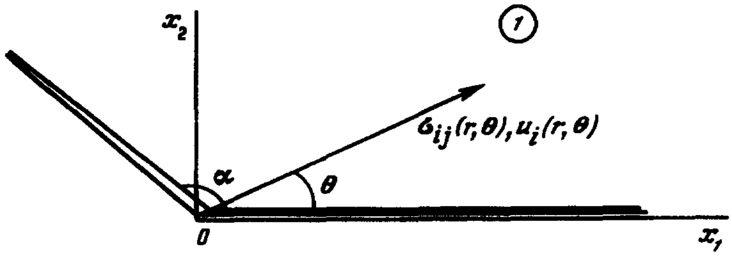

Fig. 1.

where $\sigma^{(j)}_{\gamma\beta}(r, \theta)$, $u^{(j)}_{\gamma}(r, \theta)$ ($\gamma, \beta = r, \theta, j = 1, 2$) are the components of the stress tensor and the displacement vector in the polar system of coordinates for the two different materials and ρ is the friction coefficient. Moreover, the solution of the boundary-value problem must satisfy the additional conditions

$$\sigma^{(1)}_{\theta\theta}(r,0) = \sigma^{(2)}_{\theta\theta}(r,2\pi) \leqslant 0, \quad u^{(1)}_{\theta}(r,\alpha) \leqslant u^{(2)}_{\theta}(r,\alpha) \tag{1.4}$$

being the compression condition at $\theta = 0$ and the opening condition at $\theta = \alpha$, respectively.

In the third version, the crack surfaces are in contact along both sides of the corner point and the boundary-value problem has the form

$$\begin{gathered} u^{(1)}_{\theta}(r,0) = u^{(2)}_{\theta}(r,2\pi), \quad u^{(1)}_{\theta}(r,\alpha) = u^{(2)}_{\theta}(r,\alpha) \\ \sigma^{(1)}_{\theta\theta}(r,0) = \sigma^{(2)}_{\theta\theta}(r,2\pi), \quad \sigma^{(1)}_{\theta\theta}(r,\alpha) = \sigma^{(2)}_{\theta\theta}(r,\alpha) \\ \sigma^{(1)}_{r\theta}(r,0) = \rho\sigma^{(1)}_{\theta\theta}(r,0)\frac{(u^{(1)}_{r}(r,0) - u^{(2)}_{r}(r,2\pi))}{|(u^{(1)}_{r}(r,0) - u^{(2)}_{r}(r,2\pi))|} \\ \sigma^{(1)}_{r\theta}(r,\alpha) = \rho\sigma^{(1)}_{\theta\theta}(r,\alpha)\frac{(u^{(1)}_{r}(r,\alpha) - u^{(2)}_{r}(r,\alpha))}{|(u^{(1)}_{r}(r,\alpha) - u^{(2)}_{r}(r,\alpha))|} \\ \sigma^{(1)}_{r\theta}(r,0) = -\sigma^{(2)}_{r\theta}(r,2\pi), \quad \sigma^{(1)}_{r\theta}(r,\alpha) = \sigma^{(2)}_{r\theta}(r,\alpha) \end{gathered} \tag{1.5}$$

The additional conditions (the compression conditions both at $\theta = 0$ and at $\theta = \alpha$) have the form

$$\sigma^{(1)}_{\theta\theta}(r,0) = \sigma^{(2)}_{\theta\theta}(r,2\pi) \leqslant 0, \quad \sigma^{(1)}_{\theta\theta}(r,\alpha) = \sigma^{(2)}_{\theta\theta}(r,\alpha) \leqslant 0 \tag{1.6}$$

The boundary conditions (1.1)–(1.3), (1.5) with the additional conditions (1.4) and (1.6) define the asymptotic form of the solution close to the corner point, taking into account formation of contact zones and opening zones.

2. THE ASYMPTOTIC SOLUTION

In each of the characteristic domains, we write the stress–strain displacement state using the well-known complex representations [6]

$$\begin{gathered} \sigma^{(j)}_{rr} + \sigma^{(j)}_{\theta\theta} = 2[\varphi'_j(z) + \overline{\varphi'_j(z)}] \\ \sigma^{(j)}_{\theta\theta} - \sigma^{(j)}_{rr} + 2i\sigma^{(j)}_{r\theta} = 2[\bar{z}\varphi''_j(z) + \psi'_j(z)]e^{2i\theta} \\ 2\mu_j \cdot (u^{(j)}_r + iu^{(j)}_{\theta}) = e^{-i\theta}[\varkappa_j\varphi_j(z) - z\overline{\varphi'_j(z)} - \overline{\psi_j(z)}] \end{gathered} \tag{2.1}$$

where $j = 1, 2$, $\varkappa_j = 3–4\nu_j$ for plane strain and $\varkappa_j = (3 - \nu_j)/(1 + \nu_j)$ for plane stress, ν_j is Poisson's ratio μ_j is the shear modulus, and $z = re^{i\theta}$, $\varphi_j(z)$ and $\psi_j(z)$ are the complex Kolosov–Muskhelishvili potentials in the corresponding domains.

We write the solution of system (2.1) in each of the characteristic domains in the form

$$\varphi_j(z) = a_j z^\lambda, \quad \psi_j(z) = b_j z^\lambda \tag{2.2}$$

Here, a_j and b_j are unknown complex constants, and λ is assumed to be real number. In this case, expressions (2.1) will define the stress and strain fields. This is associated with the fact that, when there is a contact zone, there is no oscillating solution [7] for the asymptotic form at the interface of two materials [8]. The correctness of this assertion will be verified numerically when finding the roots of the characteristic equation.

We introduce the notation

$$a_j = A_j + iB_j, \quad b_j = C_j + iD_j, \quad j = 1, 2$$

Using relations (2.1) and (2.2), we obtain expressions for the stresses and displacements in the characteristic domains ($j = 1, 2$)

$$\sigma_{r\theta}^{(j)} = \lambda r^{\lambda-1}[A_j(\lambda-1)\sin(\lambda-1)\theta + B_j(\lambda-1)\cos(\lambda-1)\theta + C_j\sin(\lambda+1)\theta + D_j\cos(\lambda+1)\theta]$$

$$\sigma_{\theta\theta}^{(j)} = \lambda r^{\lambda-1}[A_j(\lambda+1)\cos(\lambda-1)\theta - B_j(\lambda+1)\sin(\lambda-1)\theta + C_j\cos(\lambda+1)\theta - D_j\sin(\lambda+1)\theta]$$

$$\sigma_{rr}^{(j)} = \lambda r^{\lambda-1}[A_j(3-\lambda)\cos(\lambda-1)\theta - B_j(3-\lambda)\sin(\lambda-1)\theta - \tag{2.3}$$

$$- C_j\cos(\lambda+1)\theta + D_j\sin(\lambda+1)\theta]$$

$$u_\theta^{(j)} = \frac{r^\lambda}{2\mu_j}[A_j(\lambda+\varkappa_j)\sin(\lambda-1)\theta + B_j(\lambda+\varkappa_j)\cos(\lambda-1)\theta + C_j\sin(\lambda+1)\theta + D_j\cos(\lambda+1)\theta]$$

$$u_r^{(j)} = \frac{r^\lambda}{2\mu_j}[A_j(\varkappa_j-\lambda)\cos(\lambda-1)\theta - B_j(\varkappa_j-\lambda)\sin(\lambda-1)\theta - C_j\sin(\lambda+1)\theta + D_j\cos(\lambda+1)\theta]$$

The problem therefore reduces to searching for the unknown constants A_j, B_j, C_j, D_j ($j = 1, 2$) and the value of λ. The values of the exponent λ are sought in the energy class of solutions $\lambda \geqslant 1/2$.

Substituting the explicit expressions for the stresses and displacements in domain 1 and their analogues in domain 2 into boundary conditions (1.1)–(1.3) we obtain a system of equations which are homogeneous in $A_1, B_1, C_1, D_1, A_2, B_2, C_2, D_2$. The system was solved numerically using the following algorithm: a numerical calculation of the roots of the characteristic equation by the method of secants; calculation of the unknown constants (solution of the system using Gauss' method); calculation of the stress and displacement fields and numerical analysis of the results.

It should be noted that the solution of the characteristic equation is represented by the set of values of the exponent λ: $\lambda_1, \lambda_2, \lambda_3$. The minimum value of λ, which satisfies the constraint $\lambda \geqslant 1/2$, was chosen as the index of the singularity of the solution, and the stress and displacement distributions obtained when this was done satisfied the system of inequalities (1.4) which actually determine the solutions for a crack with a contact zone.

The stress–strain state close to the corner point of a crack with a contact zone was analysed using the following algorithm. First, in accordance with conditions (1.4), it is necessary to assume that there is compression along the θ axis in the zone of contact of domains 1 and 2. Consequently, it is possible to establish a correspondence between the fully determined sign of the constant and each value of the kinking angle, to within the accuracy with which the stress $\sigma_{\theta\theta}^{(1)}$ is determined. This, in turn, enables us to determine the relative magnitude and direction of the displacements $u_\theta^{(1)}$ and $u_\theta^{(2)}$ at $\theta = \alpha$, that is, on the surface which is free according to the formulation of the problem, and to validate the second inequality of (1.4). Furthermore, we note that, when there is friction between the contacting surfaces, the asymptotic form was calculated both for the positive as well as for the negative difference between the tangential displacements of the crack surfaces (the first equation of (1.3)) and, in the case of the stress–strain state constructed on the basis of it, it was verified that it corresponded to the chosen sign. This enables us to choose the least exponent λ which corresponds to a solution of the contact problem.

The solution of boundary-value problem (1.5) with the additional conditions (1.6) was obtained in a similar manner.

3. RESULTS OF THE CALCULATION

The results of the calculation of the exponent of the asymptotic form, obtained within the framework

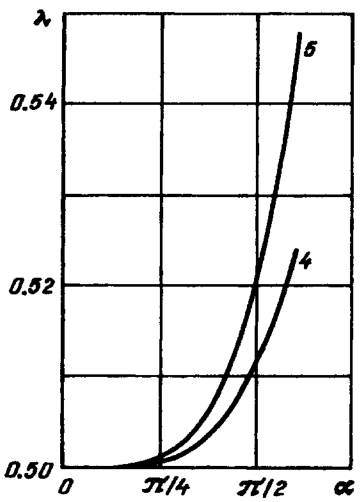

Fig. 2.

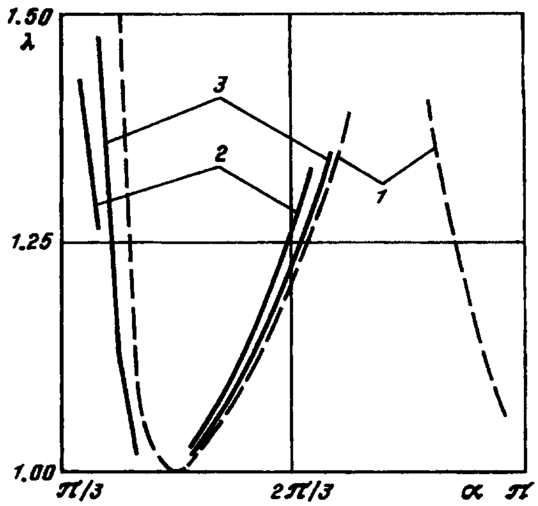

Fig. 3.

of the solution of boundary-value problem (1.1)–(1.3) subject to conditions (1.4) being satisfied, that is, for the contact ($\theta = 0$)—opening ($\theta = \alpha$) problem, are presented in Figs 2 and 3. Figure 2 shows $\lambda = \lambda(\alpha)$ for various values of the friction coefficient. Curve 4 was drawn for $\rho = 0.1$ ($\mu_1 = \mu_2$, $\nu_1 = \nu_2$) and curve 5 for $\rho = 0.2$. It can be seen that a solution of the contact—opening form is not obtained for large values of the kinking angle ($\alpha > 110°$). Figure 3 shows the values of the asymptotic form as a function of the corner angle α for different relations between the elastic characteristics of the materials in domains 1 and 2. Curve 1 is for the same materials in domains 1 and 2. Curve 2 was drawn for $\mu_1/\mu_2 = 5(\nu_1 = \nu_2)$, and curve 3 for $\mu_1/\mu_2 = 2$. At large values of the angles ($\alpha > 150°$), the values of the exponent of the asymptotic for materials with different moduli and materials with the same moduli differ only slightly by up to 5%.

Note that a change in the ratio of Poisson's ratios within the limits $1 < \nu_1/\nu_2 < 2$ has no substantial effect on the form and magnitude of the asymptotic. Conversely, a change in the ratio of the shear moduli of the materials turns out to have a substantial effect on both the nature and magnitude of the asymptotic coefficient. For instance, the case where the moduli are different leads to a definite range of corner angles (close to $\alpha = 90°$), a solution of the contact-opening type is not realized. Moreover, an increase in the ratio μ_1/μ_2 leads to an extension of this range. A comparison of curve 1 in Fig. 3 and curves 4 and 5 in Fig. 2 enables us to conclude that the development of friction between the contacting surfaces of a crack has a substantial effect on the magnitude of the exponent of the asymptotic and, in fact, leads to a singularity in the stresses close to its corner angle, which does not occur in the case of a smooth contact.

The results of a calculation for boundary-value problem (1.5), subject to conditions (1.6), that is, for the contact ($\theta = 0$)—contact ($\theta = \alpha$) problem are given in Fig. 4. Curves 4 and 5 show $\lambda = \lambda(\alpha)$ for different values of the friction coefficient, and curves 1 and 2 show $\lambda = \lambda(\alpha)$ for different relations between the elastic characteristics of the materials. The parameters used in the calculation were similar to those mentioned above and the numbering of the curves is also retained. The values of the magnitude of the exponent of the asymptotic when there is friction between the contacting surfaces are close to unity (Fig. 4, curves 4 and 5). However, for small values of the corner angle ($\alpha < 40°$), the stresses close to the corner point are singular while, at large values ($\alpha > 140°$), they are finite. Conversely, when there is no friction (Fig. 4, curves 1 and 2), singular stresses occur for large values ($\alpha > 110°$) of the corner angle while the stresses are finite for small values ($\alpha > 65°$). Differences in the moduli of the materials ($1 < \nu_1/\nu_2 < 2$, $1 < \mu_1/\mu_2 < 5$) turn out to have an insignificant effect on the nature of the change and the magnitude of the exponent of the asymptotic; one need only note that there is an extension of the range over which a solution of the contact–contact type cannot be obtained when the ratio μ_1/μ_2 is increased (Fig. 4, curve 2, $\mu_1/\mu_2 = 5$).

The results of solving problem (1.1)–(1.3), subject to conditions (1.4), that is, contact-opening, and of problem (1.5), subject to conditions (1.6), that is, contact–contact, together with the solutions obtained earlier [3, 4], that is, opening–opening, enables us to determine the ranges of angles over which one or other stress–strain state can occur and the value of the corresponding exponent of the asymptotic for various parameters of the problem.

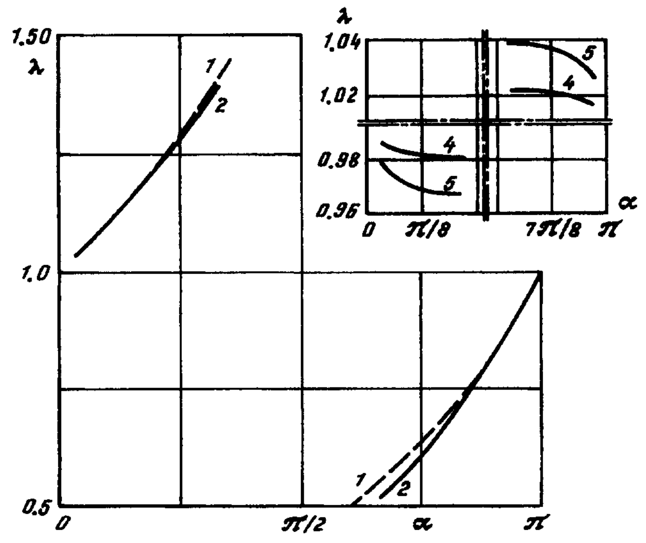

Fig. 4.

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